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THE ANALYSIS OF JOURNEY STRUCTURE: 
LINKAGE BETWEEN DIFFERENT STAGES OF 
A JOURNEY

by

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APPLICATION SUMMARY
Australian Road Research Board

THE PURPOSE OF THIS REPORT
- Is to provide a coherent basis to stimulate discussion on the possible treatment of complex journey structures on modal choice, trip generation, destination choice and activity patterns.
- To illustrate the relevance of transition matrix techniques in transport.
- To provide a start point for Project 300

THIS REPORT SHOULD INTEREST
Those active in the analysis of travel behaviour and activity structure in the disciplines of town planning, transport planning and geography.

AS A CONSEQUENCE OF THE WORK REPORTED, THE FOLLOWING ACTION IS RECOMMENDED
- Explore the existence and characteristics of complex journeys in Australian data.
- Assess the value of matrix data description methods for journey, purpose, modal choice and activity linkage purposes.
- Check to determine if Markov or semi-Markov models may be used to handle Australian transport data on a descriptive basis.
- Pursue the above inter-relationships between accessibility, behavioural models of choice, and evaluation for complex travel activities.

RELATED CURRENT ARRB RESEARCH
P300
P1058 Accessibility indicators for transport planning.

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TITLE: THE ANALYSIS OF JOURNEY STRUCTURE: LINKAGE BETWEEN DIFFERENT STAGES OF A JOURNEY.

KEYWORDS: Journey/matrix mathematical model/transport mode selection/Reading/U.K.

SUMMARY: Journeys are usually made in a number of stages involving a series of consecutive stages or trips which are linked in the sense that each stage or trip follows and precedes another stage or trip, but the decision to enter or leave a particular stage or trip may or may not be dependent upon some function of the last one entered. This report sets up tests to determine whether such 'dependent' linkages exist. One possible approach is that of Markov chains, and the use of this technique in this and other circumstances is investigated. A particular data set based on the 1971 Reading Survey has been employed to demonstrate the working of the tests, and it is shown that in this town Markov chain linkage between different stages of a journey cannot be demonstrated. The absence of such checks in other published papers using Markov approaches throws some doubt on their utility. Data on the movements of pedestrians in the Basingstoke shopping precinct have been used to show how such tests can be used in practice, and a stationary first order chain is demonstrated to be consistent with these data.


*Non. IRRD Keywords
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**CONTENTS**

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>SUMMARY</td>
<td></td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2. SOME PROBABILISTIC MODELS IN TRANSPORTATION</td>
<td>3</td>
</tr>
<tr>
<td>3. APPLICATIONS OF TRANSITION MATRICES IN TRANSPORTATION PROBLEMS</td>
<td>6</td>
</tr>
<tr>
<td>4. APPLICATIONS OF TRANSITION ANALYSIS</td>
<td>8</td>
</tr>
<tr>
<td>5. CONCLUSIONS</td>
<td>15</td>
</tr>
<tr>
<td>APPENDIX A</td>
<td>16</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>26</td>
</tr>
</tbody>
</table>
Journeys are usually made in a number of stages involving a series of consecutive stages or trips which are linked in the sense that each stage or trip follows and precedes another stage or trip, but the decision to enter or leave a particular stage or trip may or may not be dependent upon some function of the last one entered. This report sets up tests to determine whether such 'dependent' linkages exist. One possible approach is that of Markov chains, and the use of this technique in this and other circumstances is investigated. A particular data set based on the 1971 Reading Survey has been employed to demonstrate the working of the tests, and it is shown that in this town Markov chain linkage between different stages of a journey cannot be demonstrated. The absence of such checks in other published papers using Markov approaches throws some doubt on their utility. Data on the movements of pedestrians in the Basingstoke shopping precinct have been used to show how such tests can be used in practice, and a stationary first order chain is demonstrated to be consistent with these data.
1. INTRODUCTION

The modelling of travel behaviour is normally conducted at a highly aggregated level. Journeys that are undertaken in quite different ways will generally be grouped together if they have single dominant mode or purpose. If a journey is a series of linked trips to a series of destinations then each journey would probably be grouped according to major purpose and so the linkage of trips would be ignored; similarly the opportunity to model the modes of transport used during the journey is lost. Categorisation is usually by the main mode of transport employed and so the linkage with any preciding or subsequent mode of transport, destination or trip purpose is ignored.

The unit of a trip is too coarse and too ambiguous for practical use when the behavioural aspects of influencing or affecting travel demand are being considered. The coding inventions for transport data often actually ejects or aggregates the inherent link between different activities - let alone between different modes and in a single complex journey - and the effort required to reconstruct linked records of travel behaviour can be substantial, even assuming that the sequence data can be recovered from the source tapes.

Causal or behavioural analyses are always desirable, but a necessary basic first step is to assemble data in an effective and relevant structure pertinent to the behaviour in question. Such graphical and tabular devices are invaluable, and this paper pursues transition matrices as a further addition to the descriptive armoury. Certain models of aggregate behaviour are related to transition matrices of specific types, and the relevance these Markov models are examined in some detail with several examples of transport data.

If a journey is defined as a sequence of trips, then the trips are linked at first order if the probability of being on a certain trip depends only on the previous trip and not on the earlier trips of the journey. In this case, a suitable framework for recognising the influence of trips and purposes on model choice, is provided by representing each 'leg' of a multi-leg journey as a unit, and bulging transition probability matrices to summarise the probability that the following leg will be on a specific mode or purpose. Such matrices can be constructed from basic transportation survey data but are rarely produced. If trips are found to be linked then this must be recognised and suitable models for determining the pattern of but distinction and model choice must be constructed.

Non home based journeys are not as straightforward to handle as home based: there is a greater element of chance in the transfers between activities such as shopping, personal business etc., and is combined in a behavioural or causal manner can cause real problems in the proper treatment of accessibility and evaluation (Morris et al. 1978).

ACKNOWLEDGEMENTS: We would like to thank J.E. Beardwood (now of the Greater London Council) for carrying out the processing of trip data into linked journey format, and L. Gyenes, E. Dalby and J.D. Downes of the Transport and Road Research Laboratory, for access to the test data used for the illustrations in this paper. The views expressed are those of the authors, and do not necessarily reflect those of any associated organisation. The work was developed from an informal working note originally prepared in 1971 when all the named parties were working at the TRRL.
If it is desired to model modal choice using a set of transportation data, then in order to construct a pertinent model it is necessary to check to see if any relationship exists between consecutive travel movements and choices of modes of transport for these movements.

Markov processes describe systems that have a finite number of possible states, such that they are always in exactly one state at any point in time but this state may change from time to time. The process formed by the sequence of states occupied following each 'transition' may be described by a Markov chain if two fundamental conditions are satisfied.

(a) The probability of entering a certain state at a transition does not depend on the process prior to entry into the current state, and

(b) This probability does not change with time or position in the sequence of occupied states.

If the data are consistent with condition (a) an order one linkage exists; if condition (b) is satisfied an effective result for the analysis of travel choice has been determined. If condition (b) is not satisfied, the transition probability matrices derived are still a useful form in which to hold the data. If no order one linkage is found, it may still be possible to prove the existence of an order 'n' linkage. In which case condition (a) should then read:-

' The probability of entering a certain state at a transition depends only on the n-1 states occupied prior to the transition and not on the earlier history of the process'.

There are few statistical tests designed to detect the presence of Markov chains or test the inherent assumptions, notably Anderson and Goodman (1957) and Chatfield (1973). The main test used here described by Anderson and Goodman (1957) where tests are given for several cases:-

(a) That the transition probabilities for successive transitions remain constant

(b) That the chain is of a given order, and

(c) That several samples of data are from the same Markov chain of a given order.

To test each of these hypotheses Anderson and Goodman give both the likelihood ratio criterion and the \( \chi^2 \) test (see Appendix A), but they do not discuss in detail the power of the tests, nor which tests are appropriate under given circumstances.

This is covered in a paper by Lissitz (1972) where he compares the power of the likelihood ratio criterion and \( \chi^2 \) test. For the case of large samples he demonstrates that the likelihood ratio criterion has the greater power, but that the difference between the two tests is small. He concludes that either test could be used.
2. SOME PROBABILISTIC MODELS IN TRANSPORTATION

Suppose that an individual's travel movements are divided into legs, trips and journeys. Each of them may be defined in this context:

(a) A leg is that part of a trip travelled by an individual on a single mode of transport without interchange. Consequently, every time there is a change in the mode of transport a new leg is started.

(b) A trip is a sequence of legs with a common purpose; every time there is a change of purpose a new trip is started.

(c) A journey is a sequence of trips made by an individual; the last trip of a journey being one that is either not followed immediately by another trip, or is defined by some other external rule (e.g. 'return home' as a last trip in a journey).

Consider a travel data set, based on these travel units and in which there are $N$ possible modes of transport and $I$ possible purposes for each leg of a journey. From this data, one can construct a set of transition matrices which describe the pattern of transfers between successive legs of the journeys made by individuals. These matrices can be defined as follows:

For the first legs of all journeys, we have -

$$\begin{align*}
 l_{ii} & = \text{number of legs made for purpose } i, \\
 p_{i1} & = \text{proportion of legs made for purpose } i, \\
 n_{ij1} & = \text{number of legs for purpose } i \text{ made by mode } j, \\
 m_{ij1} & = \text{proportion of legs for purpose } i \text{ made by mode } j.
\end{align*}$$

Then, we develop vectors,

$$\begin{align*}
 L_1 & = (l_{11}, l_{21}, \ldots, l_{11}); \\
 P_1 & = (p_{11}, p_{21}, \ldots, p_{11}); \\
 \text{and } M_{i1} & = (m_{i11}, m_{i21}, \ldots, m_{iN1}) \quad i = 1, \ldots, I
\end{align*}$$

From each of the $M_{i1}$, form the $N \times M$ diagonal matrix,

$$M_{j1} = \begin{bmatrix}
 m_{ij1} & 0 & \cdots \\
 0 & m_{2j1} & \cdots \\
 \vdots & \vdots & \ddots \\
 0 & 0 & \cdots & m_{ij1}
\end{bmatrix}$$

Denote the set, $(M_{11}, M_{21}, \ldots, M_{N1})$ by $M_{*1}$ so that

"*" in a suffix position denotes the set over the suffix in position of the "*".


The product \( L_1 M_{*1} \) forms a vector \((l_{ij})\) of \(NI\) elements, representing the number of first legs on each mode of transport, for each purpose, i.e., \((l_{1i1}, l_{1i2}, l_{1IN}), \ldots, l_{NI1}, l_{NI2}, \ldots, l_{NN1}\).

For the second leg of all journeys, one can develop a matrix describing the transitions for each of the \((l_{ij})\) first leg elements. Expressed in probability form, this transition matrix is:

\[
P_{*1} = \begin{bmatrix}
1 & 2 & \cdots & \cdots & I \\
1 & 2 & \cdots & \cdots & I \\
1 & 2 & \cdots & \cdots & I \\
1 & 2 & \cdots & \cdots & I \\
1 & 2 & \cdots & \cdots & I \\
\end{bmatrix}
\]

where,

\[
P_{ij} = \begin{bmatrix}
1 & 2 & \cdots & \cdots & I \\
1 & 2 & \cdots & \cdots & I \\
1 & 2 & \cdots & \cdots & I \\
1 & 2 & \cdots & \cdots & I \\
1 & 2 & \cdots & \cdots & I \\
\end{bmatrix}
\]

Then the product, \( L_1 M_{*1} P_{*1} \) will form the vector,

\[
L_1 M_{*2} = \begin{bmatrix}
l_{1i1} & \cdots & m_{1i2} & \cdots & m_{1IN} \\
l_{1i1} & \cdots & m_{1i2} & \cdots & m_{1IN} \\
l_{1i1} & \cdots & m_{1i2} & \cdots & m_{1IN} \\
l_{1i1} & \cdots & m_{1i2} & \cdots & m_{1IN} \\
l_{1i1} & \cdots & m_{1i2} & \cdots & m_{1IN} \\
\end{bmatrix}
\]

where \(l_{1i1}\) \(m_{lj2}\) second legs are travelled on mode \(j\) for purpose \(i\). Similarly each of these \(l_{1i1} m_{lj2}\) can be decomposed to form a transition matrix \(P_{*2}\).

\[
L_1 M_{*3} = L_1 M_{*2} P_{*2} = L_1 M_{*1} P_{*1} P_{*2}
\]

If stationarity and a first order Markov chain can be found, then there are three matrices \(L_1, M_{*1}\) and \(P_{**}\), where \(P_{**} = P_{*1} = P_{*2} = P_{*3}\), etc., which should be sufficient to explain the pattern of legs within the first journey. The next step will be to investigate whether the pattern of legs is similar for all journey purposes, i.e. whether \(L_1, M_{*1}\) and \(P_{**}\) explain leg patterns for journey purpose 1 as well as for journey purpose 2.

In the transition matrix \(P_{**}\) or in each appropriate transition matrix \(P_{**}\) (if the process is not stationery) there will be a purpose = 'nopurpose' and a mode - 'nomode' which defines the end of a journey and represents an absorbing state. The matrices can be used to investigate the pattern of legs before absorption but not the pattern of trips. Every leg change is represented within the transition matrix but, as trip changes are implicitly included, if the pattern of trips before absorption is to be investigated a different model is needed.

Each of the \(l_{1i1} m_{lj2}\) first legs can be decomposed with respect to the mode of transport on the second legs which follow them and we can form a transition matrix, where \(l_{1i1}\) is purpose \(i\) on leg 1.
\( P_1 = \begin{bmatrix} p_{11} & 0 & \cdots & 0 \\ 0 & p_{21} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{i1} \end{bmatrix} \)

where \( P_{11} = \begin{bmatrix} p_{111} & p_{112} & \cdots & -p_{11N} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ p_{1N1} & p_{1N2} & \cdots & p_{1NN} \end{bmatrix} \)

and \( \sum_{j=1}^{n} [P_{ijn}] = 1 \)

where \( p_{i12} \) is the probability for purpose \( i \) on both legs of using mode 2 on the second leg having used mode 1 on the first leg.

The produce \( L_1 M_1 P_1 \) will be equal to the row vector.

\[
L_{11} \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1N} \\ 121 & 122 & \cdots & 12N \\ \vdots & \vdots & \ddots & \vdots \\ m_{i1} & m_{i2} & \cdots & m_{iN} \end{bmatrix} \cdots L_{11} \begin{bmatrix} m_{11} & \cdots & m_{1N} \\ 121 & \cdots & 12N \\ \vdots & \vdots & \ddots & \vdots \\ m_{N1} & \cdots & m_{NN} \end{bmatrix} = L_{11} M_2
\]

where \( \begin{bmatrix} m_{ij} \end{bmatrix} \) second legs are travelled on mode \( j \) for purpose \( i \).

Similarly each of these \( L_{11} M_{2j} \) second legs can be decomposed to form a transition matrix \( P_{*2} \).

We now have \( L_{11} M_3 = L_{11} M_2 P_2 = L_{11} M_{2*2} P_2 \)

If a Markov process is found then the matrices \( L_1, M_1, P_{*2} \) are sufficient to describe leg patterns within a trip. Trip ends have been defined by including a mode of transport ('nomode') as an absorbing state.

It is unlikely that the modes of transport used on consecutive trips will be independent, e.g., if one goes shopping driving a car, it is not usual to return home by bus.

If mode 1 is defined as being the absorbing mode, and more than \( n \) people end their first trips after \( r \) legs, then the distribution of these \( n \) people on leg \( r \) is

\[
L_{11} \begin{bmatrix} m_{1r1} & 0 & \cdots & 0 \\ \vdots & m_{1r1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & m_{1r1} \end{bmatrix} = L_{11} \begin{bmatrix} m_{1r1} & 0 & \cdots & 0 \\ \vdots & m_{1r1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & m_{1r1} \end{bmatrix}
\]

where \( \sum_{i=1}^{k} L_{11} m_{1r1} = n \)

and on leg \( r-1 \) = \( L_{11} \begin{bmatrix} m_{1,r-1,1} & m_{1,r-1,2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \cdots L_{11} \begin{bmatrix} m_{1,r-1,1} & m_{1,r-1,2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \)

where \( \sum_{k=1}^{k} \sum_{i=1}^{i} L_{11} m_{1,r-1,k} = n \)

Each of these \( L_{11} \begin{bmatrix} m_{i,r-1,1} & \cdots & m_{i,r-1,k} \end{bmatrix} \) last legs can be decomposed with respect to the modes of transport and purpose on the first leg of a second trip which follows.
Then transition matrices can be formed

and,

\[
S^{****} = \begin{bmatrix}
S_{jk11} & S_{jm12} & \cdots & S_{jk11}
end{bmatrix}
\]

where \( \sum_{r=1}^{k} S_{jrk1} = 1 \)

for each mode \( j \).

Where \( S_{jk1i} \) is the probability of going from mode \( j \) purpose 1 on the last leg of the first trip to mode \( k \) purpose \( i \) on the first leg of the second trip. Similarly a transition matrix \( S \) can be formed to express change of mode and purpose between second and third trips and a test can be applied for a unique \( S \).

Hence with the use of the matrices \( L,M,P,S \) it is possible to explain both trip and leg patterns within the first journey. Similar \( L,M,P,S \) matrices can be found for subsequent journeys and the previously mentioned statistical tests can be carried out to discover if their values are constant through all journeys. The row vectors \( L \) are, however, more likely to show some relationship akin to a skewed distribution, since the number of people who undertake journeys are unlikely to remain constant throughout a day, and indeed a further disaggregation over times of day would allow this sort of data to be used for forecasting in conjunction with a transition matrix procedure. As previously remarked, this general scheme will normally have to be modified by the initial transition matrix for first leg linkage, feeding a stationary first order Markov process for the subsequent legs only. In view of the dominance of single leg trips, this was expected to be an efficient way to handle multiple leg data in the Greater London Transport Study, and such a proposal was subsequently adopted by the GLC (reported in Heggie's (1976) summary working meeting on this subject held in 1975 at Oxford University).

3. APPLICATIONS OF TRANSITION MATRICES IN TRANSPORTATION PROBLEMS.

Linkage between trips and between legs cannot always be ignored, and that it is possible to test for its existence. If an \( n \)th order chain is detected the appropriate form of Markov process can be used to predict a pattern of future trip purposes or modes of transport on successive legs given any initial pattern of purposes or modes. It should be stressed that only if a first order or stationary process is detected can the history of each individual journey be ignored (thus the GLC approach); the \( n \)th order chain models are the only ones we have so far identified as suitable to handle the case of significant linkage along a whole chain.
The practical consequences are best described for specific examples. Data on complex journeys can be put into the form of transition matrices between sets of states. These states may be changes of mode, changes in purpose, changes in location - or indeed any combination of these or other categories. Information is lost when constructing such aggregated matrices, but once set up, the importance of any specific set of state definitions can be tested directly. If a stationary process of first order is detected, then the history before the present state can be ignored and a stable pattern of transition probabilities through time can be relied upon. These are the assumptions implicit in transportation models that break up the major legs in a complete journey and aggregate them with all other legs of the same purpose or mode and regard the modal legs as 'trips'. This might well be adequate for many purposes, but if policies depend on decisions made earlier in a chain to affect conditions later in that chain (e.g., park and ride, pedestrianisation, etc.), then it becomes important to check specifically whether or not this can be done. The method of carrying out this check may also provide the means for proceeding even if first order processes cannot be confirmed - as long as a higher order linkage is found. Stationarity is not essential, as matrices can be updated through time yet retain first order (or indeed nth order) Markov characteristics.

If a first order chain is detected, the way is clear to build simple Markov models of the process, based on this finding. If however it is discovered that the experimentally determined transition matrices do not contain a Markov chain of any order, then it is impossible to treat each state transition as being independent on the previous history in reaching that state. In other words, the proof that a Markov chain does not exist proves that the connection between the different state is not independent of the previous history of that each state transition is dependent on some complex function of the previous history. It would then be worth designing an explicit or causal model instead of a simple descriptive probabilistic Markov model.

Sasaki (1973) starts with the basic premise that people start a day of trip making from a base, e.g. their home. If the returning place of the final trip coincides with the starting place of the first, then a chain of trips is completed through a Markov process. Each trip has an associated purpose. He develops a theoretical model to determine the person trip pattern, and the effect of variation of trip purposes and zones in his modal split model. A transition matrix of trip purposes is suggested, but no steps are taken to check on the validity of the treatment of the matrix as a Markov process. The expected total number of trips with purpose j after starting with purpose i are calculated from this matrix, but more as an example than an indicator of the general pattern of trips. Sasaki concludes that the origin/destination pattern of trips described as a Markov chain will be useful in estimation procedures, but that the values of the transition matrix between purposes shown in his paper would have to be verified before use. It is essential to test for the presence of a chain before Sasaki's approach can be used in any given test.

Other authors adopting Markov approaches have also failed to deal with the statistical tests for Markov chains: Horton and Shuldiner (1967) dealt with the case of land use linkages, and developed three two-dimensional transition probability matrices, showing land use at the origin of a trip related to the land use at the destination of a trip. A first order stationary Markov chain is implicitly assumed, but no attempt is made to justify this assumption. They examined the transition matrices and drew conclusions (e.g. '92% of trip makers will be found distributed between four land uses') and proceeded to use results calculated on the basis that their data represented a first order stationary Markov process. Typical results were: 'the expected average number of stops and variances of trips with a particular land use designated as the first stop',
the expected mean number of stops per trip set, and the variance for the system as a whole', and the mean first passage time (i.e., the number of time periods it takes the trip maker to return to the state from which he started). Their paper showed the potential usefulness of Markov models in handling complex data, linked changes but made no attempt to confirm that such models were consistent with the date or processes that they were studying!

Other authors who explore the Markov approach include Gilbert (1972) who dealt with neighbourhood housing turnover, and Gilbert, Peterson and Shofer (1972) whose paper dealt with linked trip behaviour. No investigation has been carried out on the validity of using a Markov transition probability matrix to describe the system in these essentially theoretical papers and the other two mentioned earlier. Unfortunately insufficient data were published to allow a check on the validity of these basic assumptions in these published examples.

If Markov-based models of journey patterns are to be effective and valid, the existence of a detectable linkage must be statistically confirmed. If any specific results are obtained then these will determine the form of any such useful model, and must be available before any predictor model can be constructed. If linkage cannot be detected it is not possible to make any use of a deductive or synthetic model which assumes its presence.

4. APPLICATIONS OF TRANSITION ANALYSIS

The more general application of transition matrix significance tests is now considered.

The analytical schemes discussed in this paper were developed to make effective use of the class of surveys typified by the survey of shopping in Watford by the Building Research Establishment (1971), and the 1971 Reading re-survey (Downes, J.D. and Wroot, R. 1971). Both of these studies retained travel linkage information in an accessible form. Whilst a full treatment of linkages requires a consideration of purpose and mode we will first consider only trip purposes and ignore changes in the modes of transport used.

If these are I possible purposes for a trip in the data for a day's journeys by a person, then the number of first trips on each purpose can be used to form the row vector \( L = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} \) where \( 1 \) first trips (1) are travelled for purpose \( i \). In a Markovian formulation one of these I 'trip purposes' will actually represent 'no trips' so that a transition to this state signifies the end of a journey.

If there are \( N \) people in the survey then \( \sum_{k=1}^{N} 1 = N \). Each of these \( 1 \) first trips can similarly be decomposed with respect to the purpose of the second trips which follow, and expressed in the form of a transition matrix \( Y_{**1} \) where \( Y_{ij} \) is the number of people making second trips with purpose \( j \) and first trips with purpose \( i \). The product \( L*Y_{**1} \) will be equal to the row vector \( L*2 \) and where \( L*2 = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} \) as second trips are undertaken for purpose \( i \).
Similarly each of those second trips can be decomposed to form a transition matrix.

\[ Y_{2 \times 2} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \]

\[ L_3 = L_2 Y_{2 \times 2} = L_1 Y_{1 \times 1} Y_{2 \times 2} \text{ and so on.} \]

For these matrices to represent a Markov process the transition probabilities between trip purposes should be constant through time. Observed values making up these matrices can be tested by the statistical procedures mentioned previously. If necessary the level of representation of the data can be taken to a lower level of aggregated until Markov assumptions can be confirmed. Spilerman (1972) for example, makes the assumptions that:

(a) each individual or sub-population follows a Markov process and

(b) that this behaviour is represented by a separate transition matrix for each sub-population, which must be estimated from the observed-choice data by regression.

Anderson and Goodman (1957) specify tests for first order linkage by setting up transition matrices of the form

\[ Z_{n \times n} = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{bmatrix} \]

where \( z_{ijk} \) are the transition matrix elements \((ij)\) and \(k\) specifies the order of the chain linkage for a probability matrix \( Z_{n \times n} \).

The null hypothesis that the chain is of first order is tested against the alternative hypothesis that it is of second order. In the general case the null hypothesis tested is that the process is a chain of order \( n-1 \), and the alternative that it is a chain of order \( n \). The appropriate set of \( 'n' \) transition matrices are needed.

If a chain of order \( n \) cannot be detected then there is no evidence of linkage for any \( n \), and trips can be regarded as independent of any linkage. If however the process is of order \( n \) but is not stationary (i.e., the transition probabilities are not constant through time) then the model can still be used to predict the pattern of future travel transitions by retaining the different \( Z \) matrices, but a chain of order \( n \) must be established before models using transition probability matrices can be set up.
Data from a transport survey carried out in Reading (Local Government O.R. Unit 1972) are now used to illustrate the use of these tests to see if linkage between trips or legs can be ignored or if it must be taken due account of in subsequent analyses. The production of the matrices required substantial reworking of the basic data tapes, using Beardwood's generalised system of Table building and file manipulation (Millier 1974).

Four transition probability matrices were set up. Each matrix describes transitions between different purposes for up to four consecutive legs. To limit the size of this example several different purposes were aggregated into groups:

(a) Return home
(b) Go to work
(c) Go to school
(d) On work/firm's/personal business
(e) Shopping
(f) Meals/social or recreation/to take or collect another person.

The four matrices were tested for stationarity, yielding a value of chi-squared of over 36,000 for 78 degrees of freedom. This indicates that these matrices do not represent a stationary process. Later work (Havers 1975) suggests that the exclusion of the first leg of non-home-based trips permits a Markov description of the remaining legs: a finding similar to that given for shopping precinct journeys later in this paper.

The second order transition probabilities were set up in the three further matrices describing transitions between purposes for four consecutive legs. Both the likelihood ratio criterion and the \( \chi^2 \) are asymptotic and become less reliable for small numbers. It is usually (Cochran 1952) considered that \( \chi^2 \) will be accurate if no more than 5% of the observed elements are less than 5. If this is not the case further aggregation of matrix categories is needed. For the Reading data this required aggregation of trip purposes as follows:

(a) Return home
(b) Go to work/go to school
(c) On work/firm's/personal business/shopping
(d) Meals/social or recreation/to take or collect another person.

The three second order matrices were then tested for the presence of a first order chain. Anderson and Goodman's test produced a value of chi-squared of over 97,000 on 90 degrees of freedom: this clearly does not indicate the presence of a first order chain. The three second order matrices were also unsuccessfully tested for stationarity with a value of chi-squared of over 15,000 on 82 degrees of freedom. The process was not repeated for second order chains due to limitations in the programming system in use. Linkage can also be studied on a different basis. A series of matrices representing the mode of transport change between legs were set up. The first order matrix has four classes of modes of transport:
(a) drove car or light van/passenger in car or light van/taxi passenger/
drove commercial vehicle/passenger in commercial vehicle.
(b) bus passenger/train
(c) drove motor cycle or power cycle/passenger on motor cycle or power cycle/
pedal cycle
(d) walked

This matrix was tested for stationarity and the value of chi-squared of
over 2,000 on 33 degrees of freedom indicates that a stationary process does not
exist. Evidently the use of transition probability matrices is not debared,
but the assumption that a Markov process may be present is quite unwarranted.
The general indications are that the first leg of a complex journey is best
excluded if Markov models are to be adopted.

Data have been collected (Dalby 1973) on the type and sequence of shops
visited in the Basingstoke shopping precinct. In this pilot survey records
were available for 359 people who visited one or more shops in the precinct.
Some of the cells in the transition matrix were therefore based on very few
observations. Let these be I shops or groups of shops which may be visited.

The row vector \[ S_{*1} = [s_{11}, s_{12}, \ldots, s_{1I}] \]
can be constructed where \( s_{1i} \) is the
number of people going to shop \( i \) on their first visit to any shop. If there
\( N \) people in the survey \( \sum_{i=1}^{I} s_{1i} = N \). Each of these first visits \( s_{1i} \) can be
expressed in terms of the next shops to be visited; this will produce a further
row vector for each element of the initial vector \( S_{1} \):

\[
Y_{*1} = \begin{bmatrix}
\ldots & \ldots & \ldots & \ldots \\
\ldots & y_{111} & y_{112} & \ldots \\
\ldots & \ldots & y_{121} & \ldots \\
\ldots & y_{211} & \ldots & y_{212} \\
\ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]

A transition matrix \( Y_{*1} \) can
thus be constructed.

A transition matrix \( Y_{**} \) can thus be constructed. Where \( s_{i1} \) is
the number of people going to shop \( j \) immediately after a visit to shop of type \( i \).

\[
\sum_{j=1}^{I} y_{ij1} = 1 \text{ and } y_{ij1} \text{ is the probability of visiting a shop of type } i \text{ first and } \]
then on the shop of type \( j \). The product \( S_{**}Y_{**} \) is the row vector \( S_{**} = [s_{12}, s_{22}, \ldots, s_{1I}] \)
where \( s_{i2} \) people visit shop \( i \) on their second visit to any shop.

Each \( S_{**} \) can be decomposed to form a transition matrix \( Y_{**} \) and \( S_{2**} = S_{1**} * Y_{2} \) and
so on. Restating the assumptions underlying the use of Markov chain theory:

(a) If the probability of entering a particular shop depends only on the shop
visited prior to moving between shops and not on the earlier shops entered.
For example if shop A, shop B and shop C are visited successively, the
probability of entering shop C depends only on the shop B and not on the
fact that shop A was previously visited. This would be a first order chain. It is possible that a first order chain does not exist, and it is necessary to examine nth order chains, e.g., if shops 1, 2, 3, ..., k+1, k+2 are visited successively the probability \( (\pi_1) \) of entering shop k+2 depends only on the fact that shops \( w, 3, 4, ..., k+1 \) were visited in sequence, where

\[
\sum_{i=2}^{k+1} \pi_i = 1
\]

(b) If this probability \( (\pi_i) \) does not change with time or position in the sequence of shops entered then the process would be stationary.

The statistical tests are structured such that it is convenient to test for the validity of assumption 2, but when doing this it must be assumed that the \( Y \) matrices correctly represent the system. As these \( Y \) matrices are restricted here to first order, the test may need to be repeated if nth order linkage is found.

If it is confirmed that a stationary chain exists then \( Y_1 = Y_2 = Y_3 \) etc., and movements between certain type of shop follow an identical pattern whatever the position in the sequence of visits is observed. This deduction also depends on the existence of an nth order chain.

The existence of an nth order chain can now be tested. If a first order chain does not exist then it is possible that movements between certain shops are completely random, or depend on the complete sequence of n shops visited previously. Stationarity at second order is tested and a check made for second order linkage; these tests are repeated for third, fourth and to as high an order as there are states in the row vectors. Second order linkage requires a set of \( Y \) matrices in the form \( I \) and \( y_{ijk} \) is the probability of visiting shop type \( k \) after visiting shop type \( j \) and shop type \( i \) successively. These \( Y \) matrices would first be tested for stationarity with the modified test in Appendix A.

\[
\begin{bmatrix}
Y_{111} & Y_{112} & \cdots & Y_{113} \\
Y_{111} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

If neither linkage or stationarity is confirmed then movements between shop types can be regarded as random. The same conclusion can be drawn if stationarity but not linkage is found.

If linkage but not stationarity is proved then the movement pattern for individuals visiting different types of shops can be described, i.e., a probability can be assigned to visiting a particular shop given the shop presently occupied and the last \( n-1 \) shops entered (where \( n \) is the order of the chain). The movement pattern has been demonstrated to be unstable. This is reflected in the different \( Y \) matrices.
If both linkage and stationarity are proved then the sequence of shops visited given the first shop in the chain can be predicted. This requires the row vector $S_1$ and the matrix $Y$.

The Basingstoke data are given in incidence matrix form in Tables 1-6. The corresponding transition probability matrices are obtained by normalising each row to unity. A check for first order stationarity is made first. This is intuitively unlikely as the probability of leaving the precinct would increase as the number of shops visited increased (see Fig 1). Category 5 (the state) of being outside the precinct includes those people who leave the precinct at the end of their shopping trip, and those who leave only temporarily perhaps to deposit shopping in their car. With more data these two categories could be separated, but the numbers in our sample were too small. The movements between shop types 1 - 4 inclusive for stationarity. In several of the tables 1 - 6 show that more than 5% of the cells contained less than 5 entries. Cochran (1952) suggests that for tables with more than 1 degree of freedom and some expectations greater than 5 use $\chi^2$ without correction.

Stationarity tests were applied to tables 1 - 4 excluding the state representing entry and exit from the precinct. The $\chi^2$ value of 30.7 on 24 degrees of freedom is not a significant value, and confirms the hypothesis of first order stationarity. Three further tests for stationarity were applied. The first was on shop types 1, 2, 3, 4.* Only one cell value was less than 5 and chi-squared was 22 on 18 degrees of freedom. This also is not significant and is consistent with first order stationarity. For shop types 1, 2, 3, 4 no cell values were less than 5 and $\chi^2$ was 7 on 8 degrees of freedom. This too is consistent with a first order stationary process. As the entry and exit cells had been excluded, the stationarity the entry and exit to the precinct cells was checked by testing tables 1, 2, 3 and 4 with shop types 3 and 4 combined (i.e., 3,4)* a $\chi^2$ of 59 on 33 degrees of freedom is significant at the 99% level, and indicates that the entry and exit processes are not stationary at any order, and indeed probably belong to a different process: a commonsense conclusion.

Movement between shop types 1, 2, 3, 4 and 5 was tested for first order to give a $\chi^2$ of 26.7 on 21 degrees of freedom which is fully consistent with a first order stationary Markov chain. The associated transition probability matrix is

<table>
<thead>
<tr>
<th>From</th>
<th>To 1</th>
<th>To 2</th>
<th>To 3,4</th>
</tr>
</thead>
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<td>1</td>
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<td>.21</td>
</tr>
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<td>2</td>
<td>.23</td>
<td>.52</td>
<td>.25</td>
</tr>
<tr>
<td>3,4</td>
<td>.25</td>
<td>.37</td>
<td>.38</td>
</tr>
</tbody>
</table>

A further aggregation of shop types was made between shop types 2, 3 and 4 and tested for first order. $\chi^2$ was 3.9 with 12 degrees of freedom. This is not significant.

*3,4 Denotes a single state.
The associated transition probability matrix is

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>1</th>
<th>2,3,4,</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.22</td>
<td>.38</td>
<td></td>
</tr>
<tr>
<td>2,3,4,</td>
<td>.16</td>
<td>.54</td>
<td></td>
</tr>
</tbody>
</table>

These stationary first order transition probability matrices can now be used. The initial state vector driving the first order Markov process is constructed from the observed data as follows:

The probability that the first shop visited is type 1 is .36
" " " " " " " " " " " 2 is .31
" " " " " " " " " " " 3 or 4 is .33
" " " " " " " " " " " 5 is zero

These four probabilities can be combined into an initial probability vector (.36 .31 .33; 0). It can now be deduced that having started in shop type 1 it will take on average \( 1/0.36 = 2.78 \) shops before shop type 1 is visited again.

The initial probability vector is multiplied by the transition matrix to obtain the probability vector (0.30, 0.45, 0.34) establishing the probabilities that the shopper is in any of the shops for the second shop visited. As it has been shown that the transition probabilities remain constant over time, the probability that the shopper will be in shop type \( X \) when the third shop of their visit to the precinct may be established by multiplying the new probability vector by the matrix of transition probabilities; once they have visited shop type 2, they are more likely to visit shop type 2 again, and the probability of visiting shop type 2 after having visited one shop is .44. But the probabilities do not vary greatly so too great an emphasis should not be based on these deductions. It is possible to establish whether or not there exists an equilibrium or balance such that the probability of being in any shop remains constant. We can then form the transition probability matrix:

\[
PN = \begin{bmatrix}
0.28 & 0.44 & 0.28 \\
0.28 & 0.44 & 0.28 \\
0.28 & 0.44 & 0.28
\end{bmatrix}
\]

This can be interpreted by saying that in the long run food shops will be visited 28% of the time, department stores 44% and other shops 28% of the time and also if a shopper starts in a food shop then it takes an average of \( 1/0.28 = 3.5 \) shops visited before we return to a food shop. A state in a Markov chain is an absorbing state if it is impossible to leave it. Say for the sake of this example we make visiting a food shop an absorbing state then our transition matrix \( P \) would be

\[
P = \begin{bmatrix}
1 & 0 & 0 \\
0.23 & 0.52 & 0.25 \\
0.25 & 0.37 & 0.38
\end{bmatrix}
\]

Let \( Q = \begin{bmatrix}
0.52 & 0.25 \\
0.37 & 0.38 \\
1.82 & 2.34
\end{bmatrix} \) and \( N = (I - Q)^{-1} \) if \( C = \begin{bmatrix}1 \end{bmatrix} \)

Then \( NC = \begin{bmatrix}4.34 & 4.14\end{bmatrix} \)
We can conclude that starting from shop type 2 it will take 4.24 shops before shop type 1 is visited. Starting a shop type 2 the number of times shop type 3.4 is visited before shop type 1 is visited is 1.22.

Having found evidence of stationarity and a first order chain on such a small and restricted sample, there are strong indications that a Markovian analysis could be applied to the data if larger scale surveys of this type could be undertaken. It is interesting to note that precinct movement patterns can be described so simply. The values of the transition probabilities are so close that when constructing a causal model it would be reasonable to test the hypothesis that the decision to visit any shop was random before any more complex analyses were done.

6. CONCLUSIONS

Markov process models have a useful place in the analysis and representation of linked journeys in transportation planning. It is essential to test the validity of assuming that a Markov chain is embodied in the data set before it is used, and a number of different applications of Markov transition ideas to transport problems have unfortunately omitted this step. The methods for using the tests are twofold:

(a) to check on the validity of assumptions that might be made when designing surveys or building models of linked processes, and

(b) to delineate the best Markov process that might be used to describe the system if one can be found.

The shopping precinct illustration shows how both applications arise, and suggests that if the initial leg (or 'entry trip into the process') is excluded, the use of transition probability matrices for such intractable modelling issues such as park and ride, non home-based journeys, linked modal choice journeys and pedestrianisation now looks promising.

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E Richards

Report reviewed by: M G Lay
APPENDIX A

STATISTICAL INFERENCE ABOUT MARKOV CHAINS

These results are drawn from the work of T.W. Anderson and Leo A. Goodman (1957) and apply to both first and higher order chains.

1.1 ESTIMATION OF THE PARAMETERS OF A FIRST ORDER MARKOV CHAIN

The Model

Let the States be \( i = 1, 2, \ldots, m \)

and the times of observation by \( t = 0, 1, \ldots, T \)

and \( P_{ij}(t) = \text{probability of } j \text{ at time } t \text{ given } i \text{ at } t-1 \)

Consider both (a) stationary transition probabilities \((P_{ij}(t) = P_{ij} V_t)\)

(b) non stationary transition probabilities.

Let \( n_i(0) = \text{the number of individuals in state } i \text{ at } t=0 \). In this section the \( n_i(0) \) are non random, the random case is given later in Anderson and Goodman's case, but is trivial.

An observation on a given individual consists of the sequence of states at \( t = 0, 1, 2, \ldots, T \) i.e. \( i(0), i(1), \ldots, i(T) \).

From the initial state \( i(0) \) there are \( m^T \) possible sequences, representing mutually exclusive events with probabilities.

\[
P_{i(0)i(1)} P_{i(1)i(2)} \ldots P_{i(T-1)i(T)}
\]

Let \( n_{ij}(t) = \text{the number of individuals in state } i \text{ at time } t-1 \text{ and in state } j \text{ at time } t \). It can be shown that the set of \( n_{ij}(t) \) \((a \text{ set of } m^2 T \text{ numbers} )\)

form a set of sufficient statistics for the observed sequences and that

\[
n_{ij} = \frac{1}{T} \sum_{t=1}^{T} n_{ij}(t)
\]

1.2 MAXIMUM LIKELIHOOD ESTIMATES

If the process is stationary we can estimate the \( P_{ij} \)'s by maximising

\[
T \prod_{t=1}^{T} \prod_{g,j} n_{gj(t)} n_{ij} = \prod_{g,j} P_{gj} \prod_{i,j} P_{ij}
\]
w.r.t. \( P_{ij} \) and subject to \( P_{ij} > 0 \) and \( \sum_{j=1}^{n} P_{ij} = 1 \)

which gives estimates \( \hat{P}_{ij} = \frac{n_{ij}}{n_{i}} \) where \( n_{i} = \sum_{j} n_{ij} \)

If the process is not stationary \( \hat{P}_{ij}(t) = \frac{n_{ij}(t)}{\sum_{k=1}^{n} n_{ik}} \)

Asymptotic behaviour of \( n_{ij}(t) \)

It can be shown that \( \sqrt{n} (\hat{P}_{ij} - P_{ij}) \) has an asymptotic normal distribution

Tests of hypotheses and confidence regions

Tests of hypotheses about specific probabilities and confidence regions

(a) The hypotheses that the transition probabilities are constant can be tested by the null hypotheses \( H_{0} : P_{ij}(t) \forall i,j,t \) under the alternative hypothesis, estimates of transition probabilities are given by

\[ P_{ij}(t) = \frac{n_{ij}(t)}{n_{i}(t-1)} \]

The likelihood function maximised under \( H_{0} \) is

\[ L_{0} = \prod_{t=1}^{T} \prod_{i,j} (\hat{P}_{ij}(t))^{n_{ij}(t)} \]

and under \( H_{1} \), is

\[ L_{1} = \prod_{t} \prod_{i,j} (\hat{P}_{ij}(t))^{n_{ij}(t)} \]

The ratio \( L_{0}/L_{1} \) is the likelihood ratio criterion

\[ \lambda = \prod_{t} \prod_{i,j} \frac{\hat{P}_{ij}(t)^{n_{ij}(t)}}{\hat{P}_{ij}(t)} \]

where \( -2 \log \lambda \) has a \( \chi^{2} \) distribution with \( (T-1)m(m-1) \) degrees of freedom

(Note that this tests \( P_{ij}(t) \forall i \) alternatively for a given \( i \))

\[ \lambda_{i} = \prod_{t,j} \hat{P}_{ij}(\hat{P}_{ij}(t))^{n_{ij}(t)} \]

where \( -2 \log \lambda_{i} \) is \( \chi^{2} \) with \( (m-1)(T-1) \) degrees of freedom

The distribution \( P_{ij}(t) \) can be compared over \( j \) for the various values of \( t \).

The \( \chi^{2} \) test of homogeneity can be used and we calculate

\[ \chi^{2} = \sum_{t,j} n_{ij}(t-1) \left( \hat{P}_{in}(t) - P_{ij} \right)^{2}/P_{ij} \]

which has a \( \chi^{2} \) distribution with \( (m-1)(T-1) \) degrees of freedom

(Note that this tests whether the \( P_{ij}(t) \) are constant for a given \( i \))
These are two alternative tests which both test constancy of $P_{ij}(t)$ for a given $i$. They can each be summed over $i$ to give tests for all $i$.

1.3 TESTS OF THE HYPOTHESIS THAT THE CHAIN IS OF A GIVEN ORDER

Let $n_{ijk}(t)$ be the number of individuals in state $i$ at time $t-2$, in $j$ at $t-1$ and in $k$ at $t$. Also let $n_{ij}(t-1) = \sum_k n_{ijk}(t)$

Then the $n_{ijk}(t)$ $(i,j,k = 1 \ldots m; t = 2 \ldots T)$ is a set of sufficient statistics for the different sequences of states. The conditional distribution of $n_{ijk}(t)$; given $n_{ij}(t-1)$ is

$$\frac{n_{ij}(t-1)!}{\prod_k n_{ijk}(t)!} \frac{\sum_k P_{ijk} n_{ijk}(t)}{n_{ij}(t-1)}$$

the $n_{ij}(1)$ are non random

When the process is not stationary $P_{ijk}$ should be replaced by the appropriate $P_{ijk}(t)$. The maximum likelihood estimate of $P_{ijk}$ for stationary chains is $\hat{P}_{ijk} = \frac{\sum_{t=2}^{T} n_{ijk}(t)}{\sum_{t=2}^{T} n_{ij}(t-1)}$. Test by the null hypothesis $H_0: P_{ijk} = P_{2jk} = \ldots = P_{ik}$ for $j; k = 1 \ldots m$ i.e., test the hypothesis that chain is of first order against the alternative hypothesis that it is second order.

The likelihood ratio criterion for testing this hypothesis is

$$\lambda = \prod_{i,j,k=1}^{m} \left( \frac{\hat{P}_{ijk}/\hat{P}_{ijk}}{\sum_{t=1}^{T} \sum_{t=1}^{T} n_{ijk}(t)/n_{ij}(t)} \right)$$

where $\hat{P}_{jk} = \frac{\sum_{t=2}^{T} n_{jk}(t)}{\sum_{t=1}^{T} n_{ij}(t)}$ is the maximum likelihood estimate of $P_{jk}$

Under the null hypothesis $-2 \log \lambda$ has an asymptotic $\chi^2$ distribution with $m(m-1)^2$ degrees of freedom.

Another test of the hypothesis of homogeneity can be obtained by

Calculate $\chi^2_j = \sum_{i,k} n_{ij}^* (\hat{P}_{ijk}^-\hat{P}_{jk})^2 \hat{P}_{jk}$

where $n_{ij}^* = \sum_{t=1}^{T-1} n_{ij}(t)$

If the hypothesis is true $\chi^2_j$ has the usual limiting distribution with $(m-1)^2$ degrees of freedom.
calculating
\[ \lambda_j = \prod_{i,j} \left( \frac{\hat{P}_{jk}}{\hat{P}_{ijk}} \right) (n_{ijk}) \]

where the asymptotic distribution of 
\(-2 \log \lambda_j \) is \( \chi^2 \) with \((m-1)^2\) d.f.

Either \(-2 \log \lambda_j \) or \( \chi^2_j \) may be summed over \( j \) to give a test for all \( i \) with \( m(m-1)^2 \) d.f.

Consider the joint hypothesis that \( P_{ijk} = P_{jk} \) \( \forall \ i,j,k = 1 \ldots m \)

\[
\sum_{j=1}^{m} -2 \log \lambda_j = 2 \sum_{ijk} n_{ijk} \left[ \log \hat{P}_{ijk} \right] - \log \hat{P}_{jk} \left( \hat{P}_{ijk} - \hat{P}_{jk} \right)^2 / \hat{P}_{jk}
\]

\[
\chi^2 = \sum_{ijk} n_{ijk}^* \left( \hat{P}_{ijk} - \hat{P}_{jk} \right)^2 / \hat{P}_{jk}
\]

with \( m(m-1)^2 \) degrees of freedom

Generalisation for a chain of order \( r \)

\( H_0 : P_{ij} \rightarrow k_1 = P_j \rightarrow k_1 \) \( i = 1 \ldots m \) is the chain of order \( r-1 \)

The alternative hypothesis is that it is a chain of order \( r \).

The maximum likelihood estimate of \( P_{ij} \rightarrow k_1 \) is

\[ \hat{P}_{ij} \rightarrow k_1 = \frac{n_{ij} \rightarrow k_1}{n^*_{ij} \rightarrow k} \]

where \( n_{ij} \rightarrow k_1 = \sum_{t=r}^{T} n_{ij} \rightarrow k_1(t) \) and

\[ n^*_{ij} \rightarrow k = \sum_{t=r-1}^{T-1} n_{ij} \rightarrow k(t) \]

\[ \lambda_{i \rightarrow k} = \prod_{i,j} \left( \frac{\hat{P}_{j} \rightarrow k_1}{\hat{P}_{ij} \rightarrow 1} \right) (n_{ij} \rightarrow k) \]

If the null hypothesis is true then

\[ \chi^2_{i \rightarrow k} \]

where \(-2 \log \lambda_{i \rightarrow k} \) is \( \chi^2 \) with \((m-1)^2\) degrees of freedom

\[ \chi^2 = \sum_{i,j} n^*_{i,j} \rightarrow k \left( \hat{P}_{ij} \rightarrow k_1 \rightarrow k \right)^2 / P_{1 \rightarrow k}
\]

where \( \hat{P}_{j} \rightarrow k_1 = \sum_{t=r}^{T-1} n_{j \rightarrow k_1(t)} / \sum_{t=r-1}^{T-1} n_{j \rightarrow k(t)} \)

with \((m-1)^2\) degrees of freedom

Again the test can be generalised to test for all combinations \( j \rightarrow k \) by adding the \( \chi^2 \) values to give a \( \chi^2 \) with \( m^2(m-1)^2 \) degrees of freedom.
Fig 1 - Cumulative probability of leaving precinct
TABLE I

NUMBER OF FIRST SHOPS VISITED RELATED TO NUMBER OF SECOND SHOPS VISITED

<table>
<thead>
<tr>
<th>FROM</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tbody>
<tr>
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<td>0</td>
</tr>
</tbody>
</table>

NOTE: Definitions of abbreviations used in Tables 1 - 6
1. Food and drink shops = Sainsbury's, Arthur Cooper etc.
2. Department Stores = Woolworths, Marks & Spencers, W.H. Smith etc.
3. Banks, Finance, Services and Entertainments = Banks Gazette, Library, Travel Agents, etc.
4. Other = Clothing shops, discount stores, newsagents, etc.
5. Outside the precinct.

TABLE II

NUMBER OF SECOND SHOPS VISITED RELATED TO NUMBER OF THIRD SHOPS VISITED

<table>
<thead>
<tr>
<th>FROM</th>
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</thead>
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</tr>
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<td>5</td>
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</tr>
<tr>
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<td>12</td>
<td>11</td>
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<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

TABLE III

NUMBER OF THIRD SHOPS VISITED RELATED TO NUMBER OF FOURTH SHOPS VISITED

<table>
<thead>
<tr>
<th>FROM</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>13</td>
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<td>6</td>
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</tr>
<tr>
<td>2</td>
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### TABLE IV

**NUMBER OF FOURTH SHOPS VISITED RELATED TO NUMBER OF FIFTH SHOPS VISITED**

<table>
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### TABLE V

**NUMBER OF FIRST AND SECOND SHOPS VISITED RELATED TO NUMBER OF THIRD SHOPS VISITED**

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</table>

* Group such as 3,4 represent the aggregation of groups 3 and 4.
** Groups such as 9,5 represent the sequence 9 followed by 5 as an initial state.
TABLE VI

NUMBER OF SECOND AND THIRD SHOPS VISITED RELATED TO NUMBER OF FOURTH SHOPS VISITED.

<table>
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</table>
1.4 TEST OF THE HYPOTHESIS THAT SEVERAL SAMPLES ARE FROM
THE SAME MARKOV CHAIN OF A GIVEN ORDER

(It would appear that stationary chains have been assumed).

If there are \( (h) \) \( S \) first order chains there will be \( S \) estimators of
each \( P_{ij} \). Call these \( \hat{p}_{ij} \), \( h = 1 \cdots S \)
to test \( H_0: P_{ij} = \hat{p}_{ij} \) for \( h=1 \cdots S \) and \( j=1 \cdots n \)

\[
X^2_i = \sum_{h,j} n_{ij}(h) (\hat{p}_{ij}(h) - \hat{p}_{ij}(\cdot))^2 / \hat{p}_{ij}(\cdot)
\]

where \( \hat{p}_{ij}(\cdot) = \sum_{j} n_{ij}(\cdot) / \sum_{i} n_{ij}(\cdot) \)

and \( n_{ij}(\cdot) = \sum_{h} n_{ij}(h) \)

which has a \( \chi^2 \) distribution with
\( m(m-1) \) degrees of freedom

The paper actually gives the result for \( r^{th} \) order chains.

The paper by Anderson and Goodman gives alternative tests, the \( \chi^2 \)
and the likelihood ratio criteria. Before one can judge which test is the
best to use it is advisable to know and compare the power of the \( \chi^2 \) and the
likelihood ratio tests. R.W. Lissitz' (1972) comparison of the sample power
of the chi-square and likelihood ratio tests of the assumptions for
stochastic models, does just this. He states that a computer program was
used to investigate the power of both the chi-square and transformed
likelihood ratio test statistics under four sample sizes (25, 50, 75, 100)
and two percentage lengths (3 and 5).

Three test situations were investigated

1. An order zero stationary process as null hypothesis, but the
population actually specified to be order one, stationary.

2. An order one stationary process as null hypothesis, but the
population specified to be order two and stationary, and

3. An order one stationary process as null hypothesis, but, the
population specified to be order one.
A specific transition was chosen as input to the program for each of these three cases.

Two sets of alpha critical values were used for the .1, .05, .01 alpha proportions. One set was taken directly from the standard tables for chi square distribution and the other set was empirically generated from a generated distribution with small sample size. This however would be unnecessary, when large samples were being considered. When the alpha critical value was generated the likelihood ratio statistic \((-2 \ln \lambda)\) and the chi-square statistic behaved very similarly, however the likelihood ratio statistic had greater power when tabled values of the alpha critical region were used, but the differential power was more pronounced with smaller samples. Also with the larger number of stages, the two test statistics approach each other in power, as a function of sample size, at a slower rate.

This suggests even though the likelihood ratio criteria has greater power, for larger samples the differential power is small, and therefore it would not be too unreasonable to use the chi squared statistic which is easier to compute.
REFERENCES


DOWNES, J.D. and WROOT, R. (1971). Repeat survey of travel in the Reading area. TRRL Supplementary Report 43 UC.


