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Dynamic models of modal choice

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ABSTRACT

Modal competition over time is used to examine the requirements for effective and useful dynamic models of transport systems. Mathematical programming techniques have been used to study time dependent models of bus and car competition and to determine the critical characteristics for such models. A remarkably detailed representation of the interaction within the system is needed to bring out the dynamic behaviour, and the overall behaviour of these feedback models is notably robust to different objective functions. Separable or quadratic programming is however necessary if good models of both equilibrium and dynamic behaviour are required.
1. INTRODUCTION

In the study of transport planning problems comparative static models are generally used, disaggregated in various ways as appropriate. This is fully compatible with the transport and land-use data collected, as this collection is carried out over an extended period, and is normally intended to represent an 'averaged' day when results have been collected. While this approach has been notably successful in achieving a coherent understanding of the patterns of movements and activity in urban areas, it has forced us to largely ignore many of the time-dependent inter-relationships between these patterns, or to treat them qualitatively, if at all.

There are many processes in urban transport and development that are visibly affected by leads and lags in the impact of changes. A new motorway, for instance, brings a whole family of effects into play, and the land-use changes over time are at least as important as the fact they occur at all. The central unsolved dynamic problem is that of interaction of land-use and transportation decisions but experience in this area is limited, as a survey of the literature shows. A vital component of this type of dynamic problem concerns the value and relevance of dynamic and feedback mechanisms in the generally better understood area of modal choice and trip distribution. The study of time-dependent systems, is, of course, not new to the field of economics and economic systems; but in the transportation field, study of this kind of problem has been limited mainly to the classical transportation scheduling problems (Tapiero and Saliman (1); Midler (2)). Few attempts have been made to understand the full implications of modelling time-dependent transportation problems. The classical (Hitchcock-Koopmans) type of transportation problem, and the problems of vehicle scheduling and routing over time, have a special structure which facilitates building of models and algorithms. The structure of the matrices in these problems is such that a workable algorithm may be constructed with comparative ease, and this explains why a great deal of research has been applied to this particular area. But while some characteristics and techniques carry over to time dependent problems such as modal choice, consideration of the Hitchcock-Koopmans transportation problem does not give much insight as to what level of detailed representation is required to model a dynamic system. It is comparatively easy to construct an abstract mathematical structure which specifies a time dependent situation; it is quite another matter to actually produce a workable model without some knowledge of which features of the time-dependent system are critical.

A paper by Tapiero and Saliman (1) typifies recent work on the subject. They present an abstract model which provides a reasonable and complete representation of a multi-commodity transportation problem over time, but no indication of applicative experience, which would be the next stage of their work. Some indication of how such a model could be run for production use, and if possible a demonstration of results, must be produced before such a model may be regarded as anything other than tentative.

Midler (2) adopts a more effective approach to the problem of selecting an optimal combination of transportation modes over a multiperiod planning horizon, and is very successful in this specialised area. The assumptions are clearly stated, and the difficulties involved are discussed. This is a very important point, and it is easier to accept the final algorithm because of this. Unfortunately the work Midler has done does not carry over to the general case because of the special structure of his problem, although there are many similarities. Bergendahl (3) restricts his attention to the single mode problem of optimal investment in a road network: once again these limitations restrict the value of his work in a multi-modal case.

The present paper is an initial attempt to produce an approach to general time-dependent transportation analysis by addressing the
simplest and intuitively best understood transport problem where
time-dependent effects are visibly important: namely the modal
choice between private car and public bus. Lessons gained from this
study carry over to more general time-dependent systems. The paper
contains formulations, results and discussions of this study of the
dynamic modal choice problem. The objective of the work is to ascertain
what degree of representation is required to bring out time-dependent
effects, how simple the formulations could realistically be made, what
effect different optimisation criteria would have on behaviour, and to
what global factors were the equilibria most sensitive.

The problem we address requires both the multi-modal formulations
of Midler and the network investment approach of Bergendahl and can be
considered to overlap both the specialised areas considered by these
authors.

This approach is significantly different to other papers on
this type of problem; our aim is to show exactly what the key issues
are without going into unnecessary and possibly confusing detail.
We are not at this stage interested in deriving the best and most
detailed representations, except in so far as they include the factors
we found to be the most important.

There are several initial approaches to the dynamic problems
of the kind discussed here, such as Bergendahl(3), and while the
approach of this work is different to most papers, found in the
literature, the paper by Midler(2) has enough in common to be discussed
with the results here.

2. TIME-DEPENDENT SYSTEMS AND MODAL COMPETITION OVER TIME

We are concerned with constructing models of a time-dependent system
within a mathematical programming framework. A time-dependent system
is taken to be one which is dynamic, includes time-lags, time-dependent
and time-independent interactions. The problems studied here are dynamic
in the usual sense that there are a number of systems which suffer
changes that respond to a manifold of sequential decisions, but there
is the additional feature that all these systems interact, and these
interactions themselves are not necessarily dependent upon time. The
interactions studied here are fixed (but possibly at a different level)
for each time period, and act as constraints upon the system. The form
of these interactions, together with the desire to search out 'optimal'
solutions of various kinds encourages the use of generalised optimising
frameworks such as those of mathematical programming (linear, quadratic
or separable programming) methods. Such methods may be employed to
determine where and what degree dynamic effects show up in the
representations of the system; the aim being to search out the simplest
modelling formulations capable of responding to and adequately
representing both cross-sectional and time-dependent behaviour.

The situation under study is intermodal competition in a
transport system, where a number of modes are operating in competition
with one another with each service being controlled by an operator
with specified objectives. It is the classical situation where a modal
operator tries to vary fares and services in order to achieve pre-
determined policies and budgets. Each such operator does the same
over different time scales, and hence multiple time-dependent responses
occur. Several of these are lagged as neither of the variations
mentioned above need be instantaneous, nor any traveller reactions to
the changes. Modal choice is of course dependent upon many factors,
including fare, frequency, journey and waiting times, and any special
attitudes shown towards each mode. The operator's policies are based
partly on these considerations, and partly on a pre-determined budget.
The modal split situation also involves an interaction which is not
dependent upon time. This arises, for instance from road or track
congestion; more traffic giving longer journey times.

The question now arises, how does this system behave over time
when there are two or more modes in competition? This paper presents as a first step three possible models using the simplest reasonable representations that are consistent with linear forms and linear programming methods. The numerical values of the results are of no major interest. The essential questions relate to the form and style of solutions; whether an equilibrium over the system exists at all, which constraints are active in determining solutions, and what is the effect of varying certain coefficients and constraints.

3. NOTATION

It is convenient at this stage to introduce some basic notation.

- \( i \) = mode
- \( y \) = time period
- \( n_y^i \) = frequency of mode \( i \) for time period \( y \) in number of modal units per week
- \( z_y^i \) = number of passengers on mode \( i \) for time period \( y \) per week
- \( m \) = number of modes \((m \geq 2)\)
- \( I_Y^{m+1} \) = number of non-travellers for time period \( y \) per week
- \( q \) = flow on link
- \( C_i \) = cost of running one unit of mode \( i \) for one week
- \( t \) = time (in weeks)
- \( d \) = total demand available for travel per week
- \( T_i \) = time lag for mode \( i \) (in weeks)
- \( B_i \) = budget for mode \( i \)
- \( Y \) = total number of time periods
- \( p_y^i \) = fare during period \( y \) on mode \( i \)
- \( c_y^i \) = total cost of running one unit of mode \( i \) for time period \( y \)
- \( L_i \) = journey time of mode \( i \)

4. FORMULATIONS

4.1 Scope for linearisation

The models initially discussed are linear. The justification for building such linear models is that the distinguishing features of our problem are the time and feedback structures, and it is reasonable that a model which incorporates these structures 'correctly' will produce results representative of the longitudinal behaviour of the system.

Any formulation where interaction and budget constraints are linear in form should produce non-trivial responses as long as the time structure is properly specified: this allows us to drastically simplify the problem—with some sweeping assumptions.

Consider the inter-model situation over some fixed period of time (say, two years) with the simplest possible competition between modes for a specified type of journey—namely, a bi-modal situation such as public (bus) versus private (car). Buses and cars run over common links (roads) and hence there is a direct interaction. Network effects are not included initially. For the purpose of elucidating time-dependent effects it is considered sufficient to consider just one link providing a common path for both buses and cars between a fixed origin and a fixed destination.

The first formulation is restricted to linear representations of a few basic elements and in constructing it a further important assumption regarding the time structure is made: i.e. that each mode operator reviews his services at pre-determined times, with appropriate time lags, before any required service change may be implemented. (It is arguable whether or not this is reasonably close to the real situation). It is apparent that in making this assumption the model becomes less dynamic in form and takes on a pseudo-equilibrium appearance. Nevertheless some helpful insights into the nature of the problem may be gained, and the real importance and influence of the inclusion of time structures
in this context may be discovered through this restriction to a single lagged process.

One straightforward linear model adopts a step function as the function expressing traveller reaction to fares. This implies that the small changes in fare would not cause a traveller reaction, which may not be too far from reality but is certainly an important assumption and to introduce some variation it is therefore assumed that each mode-operator deliberately fixes the fare at each service review, perhaps in response to rises in cost and changes in patronage. Traveller reaction is therefore dependent upon frequencies and journey times, and although it is difficult to say exactly how much effect this has on the format of the model solutions, the loss of fare-reaction may not be too great at this stage. Any time-lags arising from passenger reaction are also discounted, so the only lag under consideration is that due to service change implementation. It is sufficient to study responses of the kind required here to consider just one such lag.

The constraints for this elemental problem are generated using the following set of relationships upon which travellers' modal choice depends:

(i) Frequency of service/Number of passengers.
(ii) Modal journey time/Number of passengers.
(iii) Journey time/Flow on link.
(iv) Cost of running a service/Time.

The fact that the two modes considered use the same link (and hence contribute to the flow on that link) generates the interaction constraints on the system (b) in the next section. In these models just this one type of interaction constraint is considered, since a number of such interactions may be compressed into one in the two-dimensional linear case. For each time period there are therefore just two constraints - interaction and demand; globally there is one constraint for each mode incorporating budget and traveller reaction.

The time period structure incorporated into the first formulation (subsequently referred to as the 'Simple Model') is perhaps most easily and clearly described by means of the diagram in Fig.1 (A), which shows how the total time is split up, with respect to one mode. In Fig.1(A), T is the time lag inherent in the problem, and R is the length of time between service reviews. Each of the time intervals marked in the diagram will be termed 'time-periods'. The special configuration of the problem stems from the existence of the time periods. One variable is introduced for the number of people (perhaps per week) travelling by each mode in each time period. The demand and interaction constraints hence appear a number of times down the diagonal of the matrix of the linear program. This is a basic feature of all the models - the matrices have a block-diagonal structure.

In these models linear functions for (i) and (iv) above are assumed to be of the form:

(i) \[ p_i^y = n_i^y \] \[ P_i \text{ constant} \quad i = 1, 2 \]
(ii) \[ p_i^0 L_i^0 - l_i \] \[ L_i^0, V_i \text{ constant} \quad i = 1, 2 \]
(iii) \[ q = \alpha_i + \beta_i L_i \] \[ \alpha_i, \beta_i \text{ constants} \quad i = 1, 2 \]
(iv) \[ C_i = C_i^0 + S_i \] \[ C_i^0, S_i \text{ constants} \quad i = 1, 2 \]

and for the interaction between journey times

(v) \[ \sum_{i=1}^2 a_i L_i = \varepsilon \] \[ \varepsilon \text{ constant} \]

The set of constants \( p_i, L_i^0, V_i, \alpha_i, \beta_i, C_i, S_i \), therefore specify the particular situation under study, which is here bimodal.
4.2 The simple model in mathematical terms

The constraints of the problem take the following form for a bi-modal model:

(a) \( \sum_{i=1}^{3} z_{i}^{y} = d \)  
Each \( y, 1 \leq y \leq Y \) (demand) and \( i \) runs from 1 to 3 to cover the 'no travel' mode.

(b) \( -\lambda_{1} z_{1}^{y} - \lambda_{2} z_{2}^{y} = \eta \)  
Each \( y, 1 \leq y \leq Y \) (interaction) \( \eta, \lambda_{1} \) constants

(c) \( \sum_{y=1}^{Y} (p_{i} z_{1}^{y} - n_{1} c_{1}^{y}) > B_{i} \)  
Each \( i, 1 \leq i \leq 2 \) (budget)

or explicitly

(c) \( \sum_{y=1}^{Y} (p_{i} z_{1}^{y} - n_{1} c_{1}^{y}) > B_{i} \)  
Each \( i, 1 \leq i \leq 2 \)

The objective functions take on various forms such as maximisation of total and individual profit, minimisation of total running costs, and minimisation of traveller perceived costs. All these objectives contain terms as in the budget constraints (c), so that, for example, for maximisation of total revenue the expression is

\[ \sum_{i=1}^{2} \sum_{y=1}^{Y} (p_{i} z_{1}^{y} - p_{1} z_{1}^{y} c_{1}^{y}) - \sum_{y=1}^{Y} p_{i} z_{1}^{y} z_{2}^{y} \]

(where the final term indicates that a bus company sees non travellers \( z_{1}^{y} \) in terms of lost revenue to correct for the constraint of a fixed total of travellers, the 3rd mode being 'no travel').

The particular role played by each of the variables in the formulation is perhaps best illustrated by diagram Fig.1(B). It has been assumed here, of course, that \( z_{1} \) is a continuous variable, and not restricted to integer values as it would be in practice. For large demands, however, this will cause no difficulty and errors from rounding the solution values will be small.

4.3 Discussion on the Simple Model

This model can give some useful indications of the nature of the problem. The most interesting feature is that budget constraints play a large part in determining the nature of the solutions. The criteria for feasibility or infeasibility of the linear programming structure of the model are dependent primarily upon the strength of these constraints. This is as might have been expected, but it is encouraging to have this intuition confirmed by the results. Fig. 2 gives an illustration of the behaviour common to all solutions for the simple model. Such results lead to the conclusion that the form of the solutions is dependent only upon the sensitivity of the solutions with respect to infeasibility, and hence upon budgets. Variation of the objective function brings very little response - if any - in the form of solution. The same may be said for variation of the coefficients in the interaction constraints. Further study of the stable equilibria inherent in the solutions (see for example Fig. 2), particularly in the final time periods, draws two more conclusions. First: that the time lags inherent in the problem are not having much effect on the model, suggesting that in this respect the model is not formulated correctly. Second: the meaning of the stable equilibrium to be seen in the last time periods is unclear; it is real - i.e. would it continue were more periods added, or is it merely apparent and an end-effect which only appears due to the limited number of time periods? This model was constructed with just six periods for this example; consequently, what end-effects are being exposed?
4.4 The extended model

The second model presented here, The Extended Model, is also linear and is an extension of the Simple Model in two ways:

(i) The time-period structure is redesigned in an attempt to produce a model capable of richer dynamic responses.

(ii) With such a time structure, the total time under consideration is lengthened so that more of the basic time periods are removed from possible end effects and influence.

All other features of the Simple Model are essentially retained. The basic assumptions already made still hold, so that within each time period the demand and interaction constraints remain the same, while the budgets are similar within the framework of a new time period structure.

The time structure is now designed so that the total time is split up into a number of sub-time periods of equal lengths this length being compatible with the time lags inherent in the problem. The idea of pre-determined reviews of service is abandoned; instead it is assumed that each operator reviews his service continuously, with respect to the time periods, after some initial starting point. (The time lag arising from traveller reaction is once again ignored).

4.5 Mathematical Configuration of the Extended Model

With the usual notation the constraints for a bi-modal model are:

(a) \[ \sum_{i=1}^{3} z_{1}^Y = d \]  

Each \( y, 1 \leq y \leq Y \)

(b) \[ \sum_{i=1}^{2} \lambda_{1} z_{1}^Y = n \]  

Each \( y, 1 \leq y \leq Y \)

(c) \[ \sum_{y=3}^{Y} (F_{1} z_{1}^Y - C_{1} z_{1}^Y - A_{1} z_{1}^Y) \geq B_{1} \text{ for each } i \]

(d) \[ z_{1}^Y \geq A_{1} z_{1}^Y \text{ for each } i \]

As has been stated, (a) and (b) are the same as for the Simple Model, while (c) has been redesigned with the new time period structure to take account of the new time lag \( T_{1} \) (in the examples, \( T_{1} = 2 \) for bus, 0 for car) so that each change of service depends on the situation \( T_{1} \) periods before. Along with this, some starting values are required, to the number of \( T_{1} = \max (T_{1}) \), and these are provided by the \( A_{1}^Y \) in (d).

As before the objective functions take different forms but always include a combination of terms such as appear in the left hand side of (c); so that for maximisation of total profit the objective function takes the form:

Maximise \[ \sum_{i=1}^{Y} \sum_{y=3}^{Y} (F_{1} z_{1}^Y - T_{1} - C_{1} z_{1}^Y - T_{1} - A_{1} z_{1}^Y) \]

4.6 Discussion of the results from the extended model

The results from this model follow the trend established by the Simple Model.

The effect of the type of objective function on the final solution is again fairly small, though switching between different
objective functions do give slightly more variation in the results than in the Simple Model. They do not alter the style of solution appreciably, and this remains true of all the results obtained. (See for example Figs. 3 and 4). In all cases a stable equilibrium position is clearly in evidence and the effects in the final time periods are the same. Budget constraints again play an active part in determining the forms of solutions. Fig. 3 is typical of such results illustrating the bimodal distribution over the twelve time periods (2 years) under consideration, as the budget constraints are tightened. Evidently the amount of fluctuation and disequilibrium is to a large extent dependent upon the tightness of these constraints, and how near the problem is to infeasibility. Moreover, in spite of fluctuations in distribution, there is always a stretch of time periods where distribution is stable. However, the stable positions are rather different to those of the Simple Model, and lead one to suspect that these characteristics are not a result of the type of period structure, but more likely a result of over-simplification of the problem. The practically interesting conclusion is that the limitation of resources allocated to public transport transforms the problem from one of optimising to one of finding a feasible pattern of service: at this basic level it is interesting that this type of idea is brought out by the model.

Again there is a striking end effect (see Figs. 3 and 4) this time of total modal split in the final period(s). The exact nature of the end effect - whether real or apparent - is not clear.

4.7 Decisions

As the problem under investigation contains decision making as an implicit feature, it is logical to consider a version of the Extended Model which incorporates decisions in the form of zero-one variables. This model is basically the same as the Extended Model, with the same time structure, interaction and demand constraints. The budgets, however, undergo a slight change: in each time period (after an initial interval) an explicit operator’s decision is assumed - to alter service or not. The decision hinges upon a test made on the number of travellers in previous time periods.

In the mathematical terms, therefore, this Decision Model retains constraints (a), (b) and (d) of the Extended Model, but (c) takes on a slightly different form and another set of constraints (c’1), are added:

\[
(c') \sum_{t=0}^{T-1} (F_t B_t Z_t - C_t Z_{t+1} B_{t+1}) \geq B_t (1 - Y_{t+1}^T)
\]

where \(Y_{t+1}^T\) are zero-one, i.e.

\[
Z_t - Z_{t+1} \leq Y_{t+1}^T \leq Z_t - Y_{t+1}^T \leq 0
\]

where the \(Y_{t+1}^T\) are zero-one. i.e.

\[
Z_t - Z_{t+1} \leq Y_{t+1}^T \leq Z_t - Y_{t+1}^T \leq 0
\]

Objective functions are as in the Extended Model, and all other assumptions and characteristics are the same.

4.8 Discussion on the results

The results produced from this model have tended to back up ideas suggested by the previous models. Fig. 5 is an illustration of a typical solution of this Decision Model; the features are similar to the Extended Model, except that mode switching is evident even well away from the end effect, which resemble those observed with the Simple Model. A similar type of end effect is present as well. It seems therefore that including a decision-making machinery in this type of linear model does not produce results that are significantly better than those from a straightforward continuous
linear model. It is interesting to note that this echoes the similar results found in research project portfolio evaluation and optimisation models. Freeman and Gear (4) have also shown how by adroit formulation, integer programming techniques may be avoided.

4.9 Discussion of the formulations

The three models presented so far have been constructed with the expressed purpose of utilising established LP/MP algorithms, thus avoiding the time consuming task of developing suitable computer programs. It is acknowledged that this approach has required some sweeping assumptions to be made; however similar assumptions have to be made with any attempt to model time structured situations. The Simple Model contains the most basic simplifications, particularly with regard to time structure; it is apparent from results that this is indeed a gross over-simplification, but the attempt gives confirmation to what has been assumed in literature elsewhere. Namely that the time structure must be explicit.

The Extended and Decision Models are formulated similarly - if more simply than many other dynamic models, such as multi-period multi-mode transportation models. The only factor of the models presented here which lends them a special flavour is the interaction inherent in the mixed mode competition situation, but this is merely an added constraint. Consider Midler's (2) formulation for a stochastic multi-period multi-mode transportation model. Excluding the interaction factor, this model and the Extended Model are basically the same, though the latter is rather simpler. Our control variables are the variables \( Y_i^y \), the number of people travelling by mode \( i \) in period \( y \). The state variables in the Extended Model refer to demand for travel, budget levels, etc., which we have assumed fixed. As consideration here is of a single link, auxiliary and conservation equations do not appear. It is evident therefore that lessons gained from these models apply to many other dynamic situations. It cannot be denied that formulations such as Midler's have much more to offer in terms of a more precise representation of the real situation and the value of such models should not be underestimated. It may be contended that many such detailed approaches would produce equally good results if they were somewhat simpler and our models and results indicate that this could be true in many dynamic situations. Here, for example, a mixed mode transport model has been formulated succinctly while retaining the essential features of the system. The formulations presented here may be regarded as an essential stage in the process of representing dynamic systems; a model which may be regarded as the next stage in this process is presented later. The construction of a complicated formulation is readily achieved, but the elimination of unnecessary detail by practical studies of both the system and the model's behaviour is required if the results are to be comprehensible and readily appreciated. Dynamic systems exhibit a resilience and robustness that make them particularly awkward to construct, calibrate, and analyse.

5. RESULTS OF THE THREE LINEAR MODELS

The three models presented so far are linear in form; it was expected originally that such linear models, even if not numerically accurate, would at least produce results that were non-trivial and follow patterns that might be observed from the real situation. However, although results may be non-trivial in the form of educational content, these models do not produce the dynamic characteristics that might be expected from their internal structure. It is true the time structure incorporated into the Extended and Decision Models have a more realistic effect than the very constricting structure of the Simple Model, but even then it is only too easy to obtain stable equilibrium positions throughout the solution. Although this may be due in part to the initial basic assumptions that have been made throughout regarding the use of a single link, factors affecting passenger reaction and the like, there is no doubt that the inflexibility of a linear two-dimensional model is a major disadvantage. It is possible to see how this might be so. Consider, for example, Fig. 6 which shows the feasible region for a sub-time-period when the
The total problem is broken down into a number of sub-problems, one for each period. For each such sub-problem the model attempts to find a solution inside the demand and on the interaction constraints with regard for budgets and objectives. This is an extremely limiting situation and helps explain why the use of more sophisticated MP methods, such as integer programming with zero-one variables, does not necessarily improve results of these models by a significant amount. In the two-mode case there are only two degrees of freedom and this inhibits the success of the model to a very great extent. It may be said that the modelling of such a dynamic problem correctly requires a high degree of sophistication in the specification of the problem to be indicated in the formulation (and hence by implication perhaps more sophisticated MP methods). The non-linearities and complex constraints inherent in the time staged system are not adequately reflected in the simplification required for linear and simple mixed integer formulations.

It is not easy to foresee that this would be the case, but the linear models do show this up. The Extended and Decision Models produce more meaningful results than the Simple Model simply because they include more details of the problem. But even then variations of coefficients bring little change to the form of solutions, in fact these linear models are remarkably insensitive to variations in the objective functions, due to their inflexibility. These models are clearly inadequate for direct use, but they do help to spotlight the difficulties and crucial points in modelling this type of problem with MP methods. More realistic models would include more modes to increase flexibility and non-linear functions as both constraints and objectives. A non-linear model using separate programming would enable interactions, fares, and demand curves to be included in a straightforward manner. Such a model will be presented later in this paper.

It is necessary to remark at this point of the achievement of feasible solutions in these models. The switching effects between modes which may be interpreted in a dynamic framework are largely dependent upon the feasibility of the specific problem, and are therefore mainly controlled by the budget constraints. This explains why variation of the objective function does not significantly alter results, while variation of the budget does produce a marginal effect. The budget constraints incorporate the time lag structure in terms of sizes of coefficients in a similar way to the objective function; thus an optimal solution may be viewed as an equilibrium obtained from the forward lags 'rebounding' from the time horizon. However, in the linear bi-modal case the optimal solution is not far removed from an initial feasible state, and this is one of the major disadvantages of formulating a dynamic situation in linear terms.

It is apparent that the number of time periods employed in any dynamic model must be large enough to allow end effects to die down. The models here expose the common and basic problem of end effects in any time-dependent model with an explicit or implicit time horizon. It is interesting to speculate whether the extent of the period and level of stable modal split in all these models would alter if further time periods were to be added. There is nothing to suggest what the effect would be, except that the Extended and Decision Models produce significant end effects suggesting that some re-distribution is being forced into the final few periods. Any investigation into end effects must be carried out with the type of time-lag and time-structure in mind; in the approach for this paper each problem is solved globally as an LP problem, so that optimisation is carried out without any special regard for any particular variable or time period. This means that the time structure and decision-making machinery has to be specified completely in every model. It is possible that a better method would be decomposition of the problem. Then the sub-problem for each time-period could be solved separately, with each sub-problem being built on decisions from previous sub-problems, producing an iterative sequential procedure. The problem may be modelled very naturally by this method, although results should not be very different from a global solution. For large problems decomposition is an important consideration when the size of MP computer software is limited. As such a process is extremely cumbersome, the
advantages do not become apparent until the global problem becomes excessively large; this will happen when a large number of time periods are present, due to the block diagonal feature of the matrices in the mathematical programming configurations.

The results of the initial studies into modelling an interactive dynamic system by LP methods have been in some respects disappointing and attempts to model the problem using only straightforward linear methods have not succeeded. However, the reasons for the failures have been most enlightening, and more information has been gained from these failures than would have been possible if a more 'sophisticated' (i.e. detailed) model had been built at once. Certainly one of the main conclusions is that the behaviour of a modelled dynamic problem depends to a large extent upon the amount of information regarding the structure of the problem included in the model; it is not possible to include enough information in a linear bi-modal interpretation of the mixed modal competition system. But enough information is now available to give clear directions if more sophisticated models should be built, and these directions will be followed up in the final model to be presented.

The more powerful MP tools are required, and are more likely to be successful in reproducing dynamic behaviour for study. But it would not be surprising to discover that the predicted overall modal splits were much the same as those forecast by the 'simpler' models: the persistence of the 'simple' results through several levels of sophistication of the model over time is typical of dynamic models. As the general characteristics of the model formulations have now been established, the next step must be to introduce further degrees of freedom for response allowed by the use of several modes of competition; this will be demonstrated in the final section. It is already clear that this initial restricted analysis of problem characteristics, an approach that has been noticeably lacking in the literature available, has been of considerable value in understanding the results and in improving detailed representations productively.

6. NON-LINEAR APPROACH

As already discussed, results from the three linear models suggest that there are several essential features that any model of a mixed-modal competitive transport system must possess. A model has therefore been constructed which maintains the basic simplicity of the earlier models, but which in two particular directions is an advance along the lines proposed earlier. As a first step we have raised the number of competing modes to 3. This is a natural but necessary extension as it increases the number of degrees of freedom of the model. Secondly, the interaction constraint(s) are now non-linear. This is more realistic and allows us to include for example, exponential curves in a natural way. All other characteristics of the linear models are retained, even to the extent of maintaining linear budget and objective rows. It is recognised that this model is not at the production level, but it does allow one to demonstrate more readily the advantages of this more sophisticated model over the linear models. The use of general functions as constraints is a straightforward extension and need not be included for the purposes of this demonstration. The model has the following configuration for a model with three modes, using the established notation.

\[
\begin{align*}
(a) \quad \sum_{i=1}^{4} z_{1}^{y} &= d & & \text{each } y, 1 \leq y \leq Y \\
(b) \quad \sum_{i=1}^{3} \frac{\alpha_{i}}{z_{i}^{y}} &= \beta & & \text{each } y, 1 \leq y \leq Y \quad \beta \text{ constant} \\
(c) \quad \sum_{y=3}^{Y} \left( F_{i}^{y} z_{1}^{y-T_{i}} - C_{i}^{y} z_{1}^{y-T_{i}} \right) &\leq b_{i} & & \text{each } i, 1 \leq i \leq 3
\end{align*}
\]

The objective functions may take many forms, incorporating terms such as appear in (c).
The model requires the use of separate programming due to the non-linear functions. These functions are represented by piecewise linear approximation, and the algorithm employed (ICL(5)) generates further refinements of this approximation. The final problem submitted to the MP software system is thus larger than that used for the strictly linear models, and more computer time is required to run it.

Results from initial runs of this model are presented in Figs. 7, 8, 9 and 10. Figs. 7, 8, 9 show two runs with the same objectives but different objective functions, while Fig. 10 is similar but the taxi mode is unconstrained - i.e. the taxi operators are not concerned with costs. Fig. 10 does not demonstrate a typical result of the model as the unrestrained taxi costs produce an effectively bi-modal situation. However, it is interesting to observe that the resultant switching of demand between bus and car alone is similar to that obtained from the elementary linear bi-modal models.

Figs. 7, 8 and 9 are the first results which produce switching between 3 modes and time lag effects which may be expected from the real life situation, even though the time scale is rather short. It is known that any model of a dynamic problem which does not have an infinite time horizon produces results with characteristic start and end effects. The models of this study follow this trend but the 12 periods employed here in the initial problem do seem to be just enough to produce dynamic behaviour which is in line with practical experience, although comments referring to the linear models earlier still apply, and a 24-period model would show further improvements as the ratio of longest lag and total period covered was reduced.

Fig. 9 shows clearly time lag effects, particularly with regard to the taxi mode, but Figs. 7 and 8 show more dramatically these time effects, particularly when considered in conjunction with each other.

Consider Fig. 8: The objective function of bus revenue is maximised over the first 5 periods while no optimisation is attempted over the remaining 7. It is at once clear that bus passengers do not react to a great extent after the optimisation has ceased and are largely independent of the objective function. But there is an immediate drop in taxi passengers and, after a lag, the car passengers. Both modes eventually stabilise, but they take 6 periods to do so.

Fig. 7 now becomes particularly interesting as the position of Fig. 8 is reversed. Over the first 4 periods, when no optimisation is taking place, there is a great deal of intermodal switching, but when the maximisation begins a stable situation is obtained until the final period when the familiar end effect is seen. Thus behaviour has been obtained which may be expected from the real life situation, and which results from the linear models, have predicted. The introduction of a third mode and a non-linear constraint has produced this situation, and it is evident that the conclusions drawn from the simpler models were correct. True dynamic behaviour may only be obtained when enough realism has been built into the model; the minimum amount of information in this case consists of enough modes to allow flexibility and a fairly precise representation of the interactions. Having established that the earlier conclusions were justified, models of more complex situations may be built knowing how much information we must include in the formulations.

It is clear that the point has now been reached where meaningful models may be constructed, but with maximum economy of specification. Computer time may therefore be kept to a minimum, while useful results are still produced. It is easy to extend this type of model to include general network descriptions (multi-link), multiple modes and different types of (non-linear) constraints. As the model stands at present such generalisations would produce a rather large, but by no means unworkable, mathematical configuration. However the block diagonal structure of the time-feedback models employed here lends itself to decomposition and
recursive techniques; so much so that an extension to the LP400 system (ICL (5)) has been defined to incorporate user procedures and hence allow the use of these techniques. The advantages of this type of approach are clear as it allows us to solve sub-problems sequentially with corresponding reduction in the size of the individual problem that is actually handled. Indeed, this is a logical method of attack, as the structure of the sub-problems (one for each time period) are similar and the composition of each sub-problem depends on previous sub-problems, due to the time structure. Information may be carried forward from sub-problem to sub-problem as each is solved, facilitating the incorporation of time lags and time dependencies. Moreover a recycling process may easily be introduced, together with some sort of convergence test, and thus there is in effect an infinite time horizon.

For example, the inclusion of a generalised network produces the following configuration.

(a) \[ \sum_{i,j} z_{ijk}^m (y) \leq d_{ij} \quad \text{all } y; (i \leq y \leq Y) i, j \in S, k \in N \]

(b) \[ \sum_{i,j} \lambda^m z_{ijn}^m (y) \leq d_{kn} \quad \text{all } y; i, j \in S, n, k \in N \]

(c) \[ \sum_k z_{kn}^m (y) = z_{nk'}^m (y) \quad \text{all } y; n, k, k' \in N \]

(d) \[ f_k (Z (y)) = 0 \quad \text{all } y; l \in y, l \in L, \text{some } L \]

Max F(Z(1),...,Z(Y))

Here, S is the set of admissible zones, N the set of admissible nodes, y refers to time period, i, j to origin and destination zones, m to mode and n, k to nodes. Z is the vector of all Zijk and f, F are general functions. (a) therefore expresses the limit of demand between zones, (b) says that capacity dkn on link (k, n) must not be exceeded and (c) expresses continuity of flow at node n. f_k provides the various interaction, budget, etc, constraints. It is apparent that this configuration gives a precise description of the system, and as such may be used to provide analysis of the mixed model transport system.

7. CONCLUSIONS

The approach described in this paper may be applied to many other time-dependent situations, such as urban growth and delayed environmental impact problems. The work described here, in all types of models, has shown that a time-dependent problem has inherent characteristics which mean that any mathematical representation must include a certain number of basic features. The linear and non-linear model results have borne this out. However to obtain acceptable behaviour the model can in fact be simplified to a reasonably high degree, even though powerful MP tools, such as separable programming, may have to be used. The degree of representation proved necessary within the model indicates that separable programming and/or generalised upper bounds are indeed profitable lines to pursue.

A search of available literature has been made to find assistance in attacking this class of problem. Many papers have been found, such as (1), (2), which attempt to solve a similar type of problem, but while plenty of work has been done in postulating precise mathematical structures, there is very little evidence to suggest that such models have been tested on 'real' data to any appreciable extent. The studies presented in this Report have confirmed that complicated structures are not necessarily required to produce dynamic behaviour although certain basic features must be included at a fair level of detail to achieve this.
The application of the analytical framework now defined is
easier to specify than the processes reported here. The block diagonal
structure of this class of problems can be exploited to ensure that
large systems can be handled without further model development. An
extension to the LP400 system (ICL(5) has been defined to allow user
procedures — and consequently flexible model definitions — to be
incorporated. The initial areas of application have been selected; they are:-

(1) can a fare/operator policy be defined that is consistent
both with maximal user satisfaction and the operators
desire to minimise operating losses or maximise profits?

(2) do control or information collection policies exist which
can be employed to sustain public transport in the face
of competition?

(3) given (1), (2), what criteria should be used to award and
subdivide transport block grants to achieve specific user
and operator goals?

The tentative use of game theory to the restricted problem
of freight transport pricing and scheduling by Charnes, Kirby,
Littlechild and Raike(5) et al indicates that our less ambitious
approach to the more general problem is feasible. The work of
Bergendahl(3) on network investments indicates that extensions to
generalised networks are also feasible.

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